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K correlations and facet models in diffuse scattering†

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Abstract. The angular intensity distribution of radiation scattered by a wide range of random media can be accounted for by assuming effective source amplitude correlations involving modified Bessel functions K_ν. We investigate how such correlations can be derived from physical models of stochastic scattering systems, namely facet models with suitable slope distribution, and consider the cases of constant and varying r.m.s. slope or facet size. We study the pertinent photon statistics of the radiation scattered by these model systems and look for a link between K correlations and K distributions. As an application we revisit the concepts and laws of diffuse reflectance and discuss the existence of lambertian scatterers.

1. Introduction

In view of its abundant applications ranging from displays to remote sensing of aerosols or radar operating over the sea, the scattering of electromagnetic radiation by stochastic systems such as fluctuating continua, random distributions of discrete scatterers, and rough surfaces is of strong current interest (see, for example, [1]). Available experimental data include angular average intensity distribution (goniometry), angular amplitude and intensity autocorrelations (first- and second-order interferometry), and higher moments of the statistical distribution of the intensity (photon statistics). A sensible strategy in the investigation of such a diverse family of difficult problems is the search for common features and unifying mathematical descriptions of the experimental results. Indeed, such useful descriptions are found to be provided by the K correlations on the one hand and the K distributions on the other hand, which apply to a large variety of non-gaussian, non-markoffian stochastic scattering systems. Both the K correlations and the K

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distributions involve the modified Bessel functions of the second kind (Macdonald functions), $K_\nu(x)=h_\nu^{(1)}(ix)$, with appropriate index ν:

(i) The angular distribution of the average intensity of the radiation scattered by a wide range of isotropic stationary random media can be derived from K-correlated effective sources (see, for example, [2, 3]), where the first-order spatial correlation reads

$$\langle u^+(r_1)u(r_2)\rangle \propto (\rho/l)^\nu K_\nu(\rho/l), \quad \nu>0. \tag{1.1}$$

Here u denotes the stochastic amplitude at source positions $\mathbf{r}_1$ or $\mathbf{r}_2$, $\rho=|\mathbf{r}_1-\mathbf{r}_2|$, l stands for a correlation length, and the brackets $\langle \ldots \rangle$ indicate the ensemble average. Examples are given in§ 2.

(ii) The photon statistics of the radiation scattered by a variety of stochastic systems is K-distributed to a significant degree of accuracy [4], i.e. one finds the amplitude probability distribution

$$p(u)\propto (u/\sigma)^{\nu+1}K_\nu(u/\sigma) \tag{1.2}$$

with σ being a measure of the variance.

While the empirical result (1.2) was recently derived from an underlying stochastic model [4], the origin of the correlation (1.1) is not well understood hitherto. In the present paper we show how the K correlations (1.1) can be derived from physical models of stochastic scattering systems, namely 'micro-area' or 'facet' models (for example [5]). Appropriate models are: (i) non-gaussian slope distributions where the r.m.s. slope and the facet size are constant parameters (see § 3); and (ii) locally gaussian slope distributions where the r.m.s. slope or the facet size vary (see § 4). We thus learn that these models are useful in understanding the measured angular intensity distributions of the radiation scattered by a large class of stochastic systems. It is henceforth worthwhile to investigate the photon statistics of the scattered radiation belonging to these model systems and to look for possible links between K correlations and K distributions. As we show in § 5, a locally gaussian facet model with appropriate variation in the r.m.s. slope leads to the K distribution for the radiation scattered close to the axis. The corresponding photocounting distributions involve Whittaker functions. Moreover we learn that K correlations provide a realistic effective-source model for the radiometry [6] of stochastic sources or scatterers. The pertinent radiometric properties are studied in § 6, and the related question of finding a facet model for a lambertian scatterer is discussed in § 7.

2. K correlations

Let us denote the normalized first-order spatial correlation ('degree of coherence') of an isotropic stationary direct or indirect source by

$$W(\rho)=\langle u^+(\mathbf{r}_1)u(\mathbf{r}_2)\rangle[\langle|u(\mathbf{r}_1)|^2\rangle\langle|u(\mathbf{r}_2)|^2\rangle]^{-1/2}, \quad \rho=|\boldsymbol{\rho}|, \quad \boldsymbol{\rho}=\mathbf{r}_1-\mathbf{r}_2. \tag{2.1}$$

For a sufficiently large source aperture, the corresponding (far-zone) angular intensity distribution $I(\theta)$ is known to be essentially determined by the Fourier transform of (2.1) with respect to the variable $k\sin\theta$ where k denotes the wavenumber [2, 5, 7].

It has been known for a long time that the intensity distribution generated by a large variety of stochastic scattering systems is compatible with certain K correlations, viz.

$$W_\nu(\rho)=\frac{1}{2^{\nu-1}\Gamma(\nu)}\left(\frac{\rho}{l}\right)^\nu K_\nu\left(\frac{\rho}{l}\right). \tag{2.2}$$

We mention that for semi-integer values of ν the expression (2.2) simplifies [2]; e.g.

$$W_{1/2}(\rho)=\exp(-\rho/l), \tag{2.3}$$

$$W_{3/2}(\rho)=[1+\rho/l]\exp(-\rho/l), \tag{2.4}$$

$$W_{5/2}(\rho)=[1+\rho/l+\tfrac{1}{3}(\rho/l)^2]\exp(-\rho/l). \tag{2.5}$$

The case $\nu=1/2$ corresponds to a markoffian process with the well known objectionable derivative at $\rho\to 0$ [2]. The correlation

$$W_1(\rho)=(\rho/l)K_1(\rho/l) \tag{2.6}$$

provides a description of tropospheric turbulence in agreement with experimental data [8]. Bremmer [9] investigates several models of the turbulent atmosphere such as Villars' and Weisskopf's model [10, 11] of turbulent mixing involving the correlation (2.6) and Eckersley's model [12] of isolated clouds with exponentially decaying scattering density corresponding to the correlation (2.5). Tatarskii [13] favours the K correlation with the index $\nu=1/3$ for describing effects of the turbulent atmosphere. The same author reports that K correlations were already introduced by von Karman in the theory of turbulence. K correlations of index 1, 3/2, and 5/2 are found to describe the fluctuation of the dielectric constant of certain polymers as determined from goniometric measurements [3]. To summarize, we can say that K correlations with index $\nu=1/3$, 1, 3/2, and 5/2 are of particular physical interest. We plot these functions in figures 1 and 2. In the same figures we also plot the gaussian correlation

$$W_G(\rho)=\exp(-\rho^2/2l^2) \tag{2.7}$$

for comparison. This correlation is often used for its mathematical simplicity (for example [5]). We notice that it is not a member of the K correlation family. A superposition of (2.4) and (2.7) is found to fit the goniometric results for a certain polymer with two scales of inhomogeneity [3].

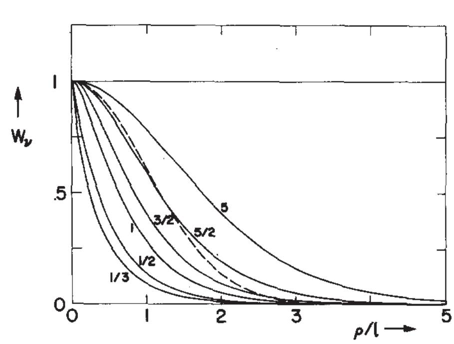

Figure 1. Linear plot of K correlations (full curves) with ν between 1/3 and 5. The gaussian correlation is shown for comparison (dashed curve).

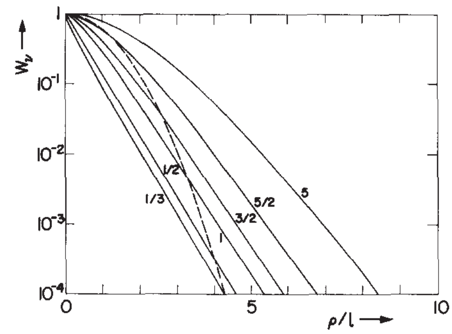

Figure 2. As figure 1, but logarithmic scale emphasizing the asymptotic behaviour for large ρ/l.

3. Facet model

We have not specified above whether the effective source correlations (2.1) to (2.7) are functions of position differences in two or three dimensions. From now on let us restrict our considerations to the two-dimensional case, i.e. planar effective sources corresponding to, for example, thin layers of stochastic scattering media. The calculations for three-dimensional effective source correlations are analogous (see, for example, [2]). Moreover we use the scalar approximation and ignore polarization effects.

In the micro-area approach the scatterer is thought of as a collection of facets of diameter ξ with a linear variation of the phase Φ, viz.

$$\Phi(\mathbf{r}) = \mathbf{mr}/\xi, \tag{3.1}$$

with $\mathbf{m}$ denoting the slope of the facet (see [1, 5, 14] and references therein). The facets are assumed to contribute incoherently to the scattered radiation field, since ξ is meant to play the role of the correlation length of the phase fluctuation produced by the stochastic medium. For instance, ξ corresponds to the correlation length of the fluctuating relief height in the case of a rough surface. The fluctuations of the scattering system are described in terms of the statistical distribution $P(\mathbf{m})$ of the slopes $\mathbf{m}$ of the facets. As an example we mention the isotropic gaussian slope distribution studied previously by Jakeman and Pusey [5], viz.

$$P_G(\mathbf{m}) = (4\pi m_0^2)^{-1} \exp(-m^2/2m_0^2) \tag{3.2}$$

of width m_0, where $m = |\mathbf{m}|$. It leads to the well known angular distribution of the scattered intensity

$$I_G(\theta) \propto \xi^2 m_0^{-2} \cos^2\theta \exp(-k^2\xi^2 \sin^2\theta/2m_0^2) \tag{3.3}$$

provided that aperture diffraction effects are neglected, i.e. $k\zeta$ is assumed to be not too small. On the other hand, the angular intensity distribution (3.3) corresponds to the gaussian correlation (2.7) with the correlation length $l = \xi/m_0$. We point out that the facet diameter or correlation length of the phase fluctuation ξ and the correlation length l of the effective source correlation are two distinct quantities.

Let us now consider a general isotropic slope distribution $P(m)$. The contribution to the angular intensity under the direction $\mathbf{s} = \sin\theta(\cos\phi, \sin\phi)$ of a facet with the phase $\Phi(\mathbf{r})$ illuminated by a uniform coherent beam reads

$$|u_F(\mathbf{s})|^2 \propto \cos^2\theta \int_F \int_F d^2\mathbf{r}_1 d^2\mathbf{r}_2 \exp[-ik\mathbf{s}(\mathbf{r}_1 - \mathbf{r}_2)] \exp\{-i[\Phi(\mathbf{r}_1) - \Phi(\mathbf{r}_2)]\} \tag{3.4}$$

with $F \propto \xi^2$ denoting the area of the facet. Using the linear phase (3.1) we obtain the contribution

$$|u_F(\mathbf{s})|^2 \propto \cos^2\theta \int_F \int_F d^2\mathbf{r}_1\, d^2\mathbf{r}_2 \exp[-ik\mathbf{s}(\mathbf{r}_1-\mathbf{r}_2)] \exp[-i\mathbf{m}(\mathbf{r}_1-\mathbf{r}_2)/\xi] \quad (3.5)$$

of a facet of slope $\mathbf{m}$. The total angular distribution $I(\theta)$ of the average intensity is obtained by averaging over the slope distribution $P(m)$. Moreover, for reasons of mathematical convenience, we account for the finite size F of the facet by considering an infinite gaussian aperture function with an effective width ξ. We thus find

$$I(\theta) \propto \cos^2\theta \int\int d^2\mathbf{r}_1\, d^2\mathbf{r}_2 \exp[-(|\mathbf{r}_1|^2+|\mathbf{r}_2|^2)/\xi^2] \times \exp[-ik\mathbf{s}(\mathbf{r}_1-\mathbf{r}_2)] \int d^2\mathbf{m}\, P(m) \exp[-i\mathbf{m}(\mathbf{r}_1-\mathbf{r}_2)/\xi]. \quad (3.6)$$

The usual transformation $\boldsymbol{\rho}=\mathbf{r}_1-\mathbf{r}_2$, $\mathbf{r}=(\mathbf{r}_1+\mathbf{r}_2)/2$ leads to

$$I(\theta) \propto \cos^2\theta \int d^2\boldsymbol{\rho} \exp(-\rho^2/2\xi^2) \exp(-ik\mathbf{s}\boldsymbol{\rho}) \times \int d^2\mathbf{m}\, P(m) \exp(-i\mathbf{m}\boldsymbol{\rho}/\xi), \quad \rho=|\boldsymbol{\rho}|. \quad (3.7)$$

This result can be rewritten in terms of a convolution, viz.

$$I(\theta)/\cos^2\theta \propto [\exp(-k^2\xi^2 s^2/2)] \otimes [P(k\xi s)] \quad (3.8)$$

with $s=|\mathbf{s}|=\sin\theta$. Neglecting aperture diffraction effects, i.e. assuming $k^2\xi^2 \gg 1$, we finally obtain

$$I(\theta) \propto \cos^2\theta\, P(k\xi \sin\theta). \quad (3.9)$$

Thus a measured isotropic angular distribution of the average scattered intensity can immediately be interpreted in terms of the pertinent micro-area model by plotting $I/\cos^2\theta$ as a function of $\sin\theta$ and by substituting the variable $\sin\theta$ by $m/k\xi$. The result (3.9) is confirmed by e.g. bubble-type projection screens showing a cut-off m_1 with $P(m)=0$ for $m>m_1$ of the slope distribution [15].

The connection with the corresponding effective source correlation $W(\rho)$ is established by recalling the well known relation (for example [6] and references therein)

$$I(\theta) \propto \cos^2\theta \int d^2\boldsymbol{\rho} \exp(-ik\mathbf{s}\boldsymbol{\rho}) W(\rho) \propto \cos^2\theta \int \rho\, d\rho\, J_0(k\rho\sin\theta) W(\rho) \quad (3.10)$$

with J_0 denoting the Bessel function of order zero. Neglecting aperture diffraction effects, comparison with (3.7) leads to

$$W(\rho) \propto \int d^2\mathbf{m} \exp(-i\mathbf{m}\boldsymbol{\rho}/\xi) P(m) \propto \int m\, dm\, J_0(m\rho/\xi) P(m). \quad (3.11)$$

Thus, not only $I(\sin\theta)/\cos^2\theta$ and $W(\rho)$, but also $P(m)$ and $W(\rho)$, are Fourier conjugates. Using the integral No. 11. 4. 44 [16], from the result (3.11) we finally learn that the K correlations (2.2) can be derived from the micro-area slope distributions

$$P_\nu(m) \propto (1+m^2/m_0^2)^{-\nu-1}. \quad (3.12)$$

The corresponding angular intensity distributions read

$$I_\nu(\theta) \propto k^2 l^2 \cos^2\theta (1+k^2 l^2 \sin^2\theta)^{-\nu-1}. \quad (3.13)$$

The facet diameter ξ, the width m_0 of the slope distribution, and the effective source correlation length obey the relation $l=\xi/m_0$.

In figures 3 and 4 we plot a number of slope distributions P_ν of physical interest. Examples of the angular intensity distributions (3.13) are shown in figures 5 and 6. In these figures we plot $I_\nu(\theta)/I_\nu(0)$ as radius vector in the relevant direction θ to obtain a polar curve or 'indicatrix' [17]. We observe that the lambertian diffuser with $I\propto\cos\theta$ is not a member of the family (3.13) and is hence not K-correlated. For comparison, we also plot in figure 5 the rather hypothetical case $I\propto\cos^2\theta$ which would correspond to the correlation $W(\rho)\propto J_1(k\rho)/k\rho$ [7] and to the slope distribution $P(m)=\text{const}$.

We finally mention that already more than 200 years ago Bouguer [18] tried to interpret his measurements of the reflectance of diffuse scatterers in terms of micro-areas, which he called *petites faces*.

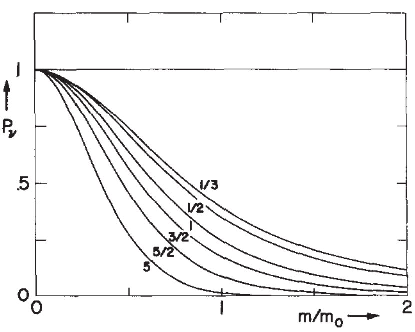

Figure 3. Linear plot of facet-model slope distribution P_ν with ν between 1/3 and 5.

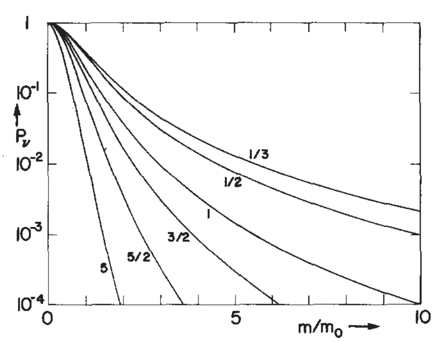

Figure 4. As figure 3, but logarithmic scale emphasizing the large slopes m/m_0.

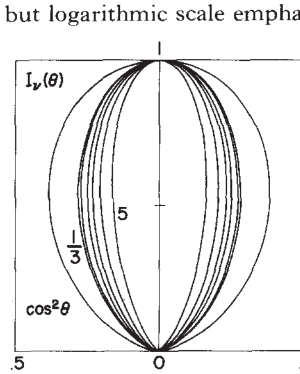

Figure 5. Polar curve illustrating the angular intensity distributions (3.13) for $\nu=1/3$, 1/2, 1, 3/2, 5/2, and 5 with constant $kl=1$. The $\cos^2\theta$ distribution is shown for comparison.

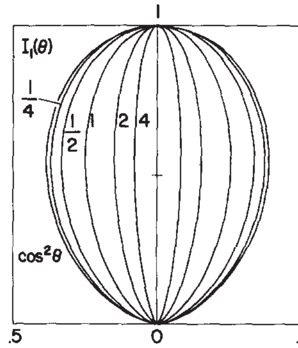

Figure 6. Polar curve illustrating the angular intensity distribution (3.13) for constant $\nu=1$ with kl between 1/4 and 4. Also shown is the $\cos^2\theta$ distribution.

4. Variations in roughness and number fluctuations

It is usually assumed that the root mean square slope m_0 and size ξ are constant parameters of the facet model. In this section and the subsequent one we investigate the consequences of variations in one or other of these quantities consistent with the slope distribution (3.12).

Consider a modified facet model in which the 'roughness' or 'texture' of the medium varies with position on a length scale which is much larger than the region of the scatterer contributing to the measured intensity. The mean intensity at an angle θ will be given by averaging equation (3.9) over the fluctuations in the slope distribution. Note that result (3.9) expresses the fact that in the geometrical optics limit, $k^2\xi^2 \gg 1$, the angular distribution of intensity is proportional to the relative frequency of finding a facet which is 'specular' with respect to the direction θ. This means that variations in the slope distribution are equivalent to changes in the *number* of contributing facets (or specular points, in the case of a smoothly changing medium) and the modified facet model therefore describes the clustering or bunching of scattering centres on a length scale larger than the region contributing to the measured intensity.

For a want of a better model we shall assume the medium to be locally gaussian with slope distribution (3.2). Variations of the parameter m_0 leading to the prescribed average slope distribution are given by the gamma distribution

$$p(x) = b^{\nu} x^{\nu-1} \exp(-bx)/\Gamma(\nu), \quad \nu > 0. \tag{4.1}$$

Averaging (3.2) over (4.1) with $x = m_0^{-2}$ leads immediately to

$$p(m) = (\nu/4\pi b)(1 + m^2/2b)^{-\nu-1} \tag{4.2}$$

which is the form (3.12) and corresponds to the angular distribution of intensity (3.13).

An interesting corollary of this result follows from the observation that in the geometrical optics limit the root mean square slope m_0 always appears in formulae as the ratio $l = \xi/m_0$ (the transformation $m \to m\xi$ expresses the slope distributions (3.2) and (3.12) in terms of this parameter). Thus instead of considering variations of m_0 with ξ fixed, m_0 could be fixed and the effect of fluctuations in ξ investigated. It has been shown previously that the simple facet model with fixed m_0 and ξ and gaussian slope distribution is closely related to a continuum gaussian phase model with gaussian phase correlation function

$$\langle\Phi(\mathbf{r}_1)\Phi(\mathbf{r}_2)\rangle/\langle\Phi^2\rangle \propto \exp\left(-|\mathbf{r}_1-\mathbf{r}_2|^2/\xi^2\right). \tag{4.3}$$

If this quantity is averaged over the distribution (4.1) with $x=\xi^2$ we obtain the K correlation function

$$\langle\Phi(\mathbf{r}_1)\Phi(\mathbf{r}_2)\rangle/\langle\Phi^2\rangle = [2b^{\nu/2}/\Gamma(\nu)]|\mathbf{r}_1-\mathbf{r}_2|^{\nu}K_\nu(2|\mathbf{r}_1-\mathbf{r}_2|b^{1/2}) \tag{4.4}$$

suggested by Tatarskii [19] as a model having the power law behaviour at small separations predicted, for example, for the spectrum of turbulence.

We have demonstrated, then, that variations in the root mean square slope of a locally gaussian facet model are equivalent, in the geometrical optics limit, to fluctuations in the number of scatterers and that this description is *consistent* with a continuum model for the scattering process characterized by a power law phase structure function.

5. Statistics of the scattered radiation

Extension of the simple facet model to calculate statistical properties of the scattered intensity is most easily accomplished using a random walk description (for example [5]). The scattered field is represented as the resultant of a two-dimensional random walk of N steps, where N is the number of contributing facets and each step measures the scattering cross-section for an individual facet. It is well known that if the number of steps is large, then the resultant field will be gaussian distributed by virtue of the central limit theorem and the corresponding intensity will be negative exponentially distributed with moments given by

$$\langle I^n\rangle = n!I^n(\theta). \tag{5.1}$$

It has recently been shown, however, that this result may not hold if the number of steps in the random walk varies, even in the limit of large mean step number [4]. It is therefore of some interest to examine the consequences for the statistics of the scattered radiation of the variations in r.m.s. slope in the last section to be equivalent to fluctuations in the number of contributing facets.

To do this we first write (3.3) in terms of the ratio l

$$I_G(\theta) \propto l^2 \exp\left(-\tfrac{1}{2}k^2l^2\sin^2\theta\right) \tag{5.2}$$

and average the nth power of this quantity over the distribution (4.1) with $x=l^2$. After normalization with the mean value of (5.2) we obtain from (5.1) ($b^{-1}=\overline{l^2}$)

$$\frac{\langle I^n\rangle}{\langle I\rangle^n} = n!\frac{\langle I_G^n(\theta)\rangle}{\langle I_G(\theta)\rangle^n} = \frac{n!\Gamma(n+\nu)}{\nu^n\Gamma(\nu)}\cdot\frac{(1+\frac{1}{2}k^2\overline{l^2}\sin^2\theta)^{n(1+\nu)}}{(1+\frac{1}{2}nk^2\overline{l^2}\sin^2\theta)^{n+\nu}}. \tag{5.3}$$

These normalized moments are plotted in figures 7 and 8. As θ increases the moments first decrease a little to a minimum at $k^2\overline{l^2}\sin^2\theta=2/\nu$ before increasing beyond their $\theta=0$ values.

Close to the axis where the angular factors can be neglected (5.3) reduces to the moments of the K-distributions (1.2). In light scattering measurements using photon-counting techniques the normalized moments are then identical to the normalized *factorial* moments of the photon counting distributions

$$p(r) = B^{\nu/2}[\Gamma(\nu+r)/\Gamma(\nu)]\exp(B/2)W_{-(r+\nu/2),(\nu-1)/2}(B) \tag{5.4}$$

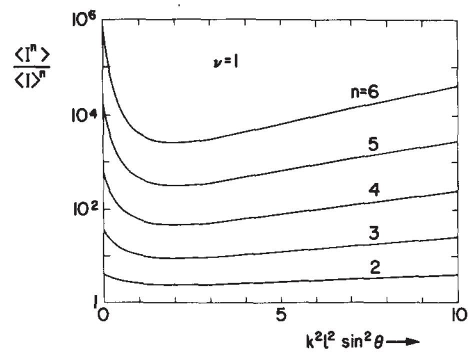

Figure 7. Plot of the second to sixth moments (5.3) for constant $\nu=1$.

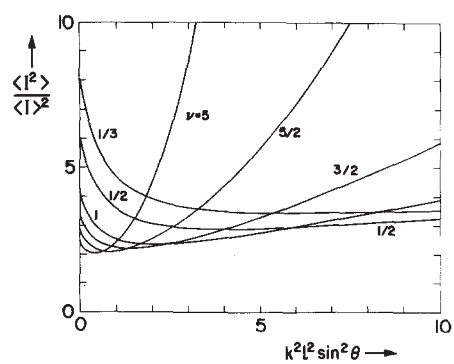

Figure 8. Plot of the second moment (equation (5.3) with $n=2$) for ν between 1/3 and 5.

where B is a constant parameter proportional to the inverse mean count rate, r is the number of photo-counts registered in a sample time short compared to the most rapid fluctuation of the light and $W_{\alpha,\beta}(x)$ is a Whittaker function [20].

It is interesting that intensity fluctuations on the axis of a laser beam which has passed through a layer or extended region of turbulent air or water are found to be K-distributed in agreement with the prediction of (5.3) (for example [21]). However, off-axis measurements on these scattering systems have not yet been made. On the other hand, result (5.3.) does not appear to be consistent with data obtained by scattering laser light from a turbulent layer of liquid crystal [22]. Intensity fluctuations are found to be accurately K-distributed even at large angles for this system [4]. Moreover, the measured angle-dependence of the second intensity moment is not of the form predicted by (5.3). The modified facet model in fact probably does not characterize the scattering by mobile systems of the turbulent type very well. For example, it does nto describe the bunching of small wavelets on a water surface effected by the modulation of a larger scale wave structure. It may well find application, however, in the scattering of radiation by static media (such as rigid rough surfaces) which have varying degree of 'roughness'.

6. Radiometry of diffuse scatterers

Diffuse reflectance or transmittance is widely used for studying samples showing a rough surface or consisting of inhomogeneous material such as packed grains or fibres. Examples are building materials, ceramics, minerals, polymerss textiles, paper, and biological material. The accepted concepts and laws of conventional diffuse reflectance [17, 23] are based on the classical theory of radiative transfer [24, 25], which deals with the propagation of intensities and neglects interference effects. As was first pointed out by Walther [7, 26] the classical concepts of radiometry and radiative transfer hold only in the hypothetical limit of strict spatial incoherence, i.e. zero correlation length, $kl \to 0$. Modern photogoniometric measurements of diffuse scatterers, however, are made using laser radiation (for example [2, 3]). Moreover, the degree of spatial coherence of a source is known to be crucial for the angular distribution of the average radiation intensity even in the case of poor coherence, e.g. $kl \lesssim 1$. Although there is much current interest in revisiting classical radiometry in the light of the theory of partial coherence (see, for example, [6]), relatively little is known on the radiometry of diffuse scatterers. Radiometric properties of random phase screens were hitherto investigated [27] only in terms of the gaussian correlation proposed by Jakeman and Pusey [5]. In view of the results presented in §§ 1–4, K correlations and the underlying models seem to provide a useful basis of further studies of the radiometry of diffuse scatterers.

We recall that the generalized radiance $B(r, s)$ can be defined as [7]

$$B(\mathbf{r}\,\bar{\mathbf{s}}) \propto \cos\theta \int d^2\rho \exp(-ik\mathbf{s}\rho)\langle u^+(\mathbf{r}+\rho/2)u(\mathbf{r}-\rho/2)\rangle, \; \mathbf{r}=(\mathbf{r}_1+\mathbf{r}_2)/2, \; \rho=\mathbf{r}_1-\mathbf{r}_2, \tag{6.1}$$

and obeys the classical relation

$$I(\mathbf{s}) = \cos\theta \int d^2\mathbf{r}\, B(\mathbf{r}, \mathbf{s}). \tag{6.2}$$

In the quasistationary situation (large diameter of the illuminating coherent beam) that we have in mind here, the micro-area model studied in § 3 leads to

$$B(\mathbf{r}, \mathbf{s}) \propto \cos\theta\, P(k\xi \sin\theta) I(\mathbf{r}) \tag{6.3}$$

with $I(\mathbf{r})$ denoting the (slowly varying) average intensity profile in the effective source plane. The corresponding generalized emittance $E(\mathbf{r})$ is defined as

$$E(\mathbf{r}) = \int \sin\theta\, d\theta\, d\phi \cos\theta B(\mathbf{r}, \mathbf{s}) \tag{6.4}$$

and reads $E(\mathbf{r}) = CI(\mathbf{r})$, where

$$C \propto \int \sin\theta\, d\theta\, d\phi I(\mathbf{s}) \propto \int \sin\theta\, d\theta \cos^2\theta P(k\xi \sin\theta) \tag{6.5}$$

denotes the radiation efficiency. This is a figure of merit for the total average energy transfer by a fluctuating scatterer as compared to that by a specular reflector, i.e. in the coherent limit (see, for example, [6] and references therein).

In the case of K-correlated scattering systems we find the generalized radiance

$$B_\nu(r, s) \propto k^2 l^2 \cos\theta (1 + k^2 l^2 \sin^2\theta)^{-\nu-1} I(r) \tag{6.6}$$

leading to the radiation efficiency

$$C_\nu \propto k^2 l^2 \int_0^{\pi/2} \sin\theta\, d\theta \cos^2\theta (1 + k^2 l^2 \sin^2\theta)^{-\nu-1}$$
$$\times \propto k^2 l^2 \int_0^1 dt (1-t)^{1/2} (1 + k^2 l^2 t)^{-\nu-1} \tag{6.7}$$

with the substitution $t=\sin^2\theta$. The integral is a representation of a hypergeometric function ([16] chapter 15). As usual we choose the normalization where $C_\nu \rightarrow 1$ in the coherent limit $kl \rightarrow \infty$. We thus obtain

$$C_\nu = \tfrac{2}{3}\nu k^2 l^2 {}_2F_1(\nu+1, 1; 5/2; -k^2 l^2). \tag{6.8}$$

We plot C_ν in figure 9 for a number of cases. For comparison we recall that the radiation efficiency of the blackbody source is $C_{BB}=1/2$ [27]. We notice that the expression (6.8) simplifies considerably for half-integer values of ν. For example we have

$$C_{1/2} = 1-(kl)^{-1}\arctan kl, \tag{6.9}$$

$$C_{3/2} = k^2 l^2 (1+k^2 l^2)^{-1}, \tag{6.10}$$

and

$$C_{5/2} = \tfrac{5}{3}k^2 l^2 (1+k^2 l^2)^{-1}[1-\tfrac{2}{5}k^2 l^2 (1+k^2 l^2)^{-1}]. \tag{6.11}$$

Another relatively simple radiation efficiency of physical interest is

$$C_1 = 1-(kl)^{-1}(1+k^2 l^2)^{-1/2}\ln[kl+(1+k^2 l^2)^{1/2}]. \tag{6.12}$$

7. Facet models and lambertian scatterers

Using the scalar approximation, Walther [7] has shown that the lambertian source with radiance

$$B_L(r, s) = \text{constant} \tag{7.1}$$

and radiant intensity

$$I_L(\theta) = \cos\theta \tag{7.2}$$

can be derived from the blackbody correlation

$$W_{BB}(\rho) = (k\rho)^{-1}\sin k\rho. \tag{7.3}$$

Let us now discuss the question whether a Lambertian radiance can also be produced by a scatterer that can be described by a facet model. In terms of geometric optics and Bouguer's classical facet hypothesis [18], this question has been studied by Grabowski [28] and Berry [29] with the conclusion that a facet distribution yielding the lambertian reflectance does not exist in the strict mathematical sense, but that the lambertian reflectance can be approximately obtained at least at small observation angles (see also [17]). We find a similar answer in terms of the modern version of the facet model as outlined in § 3.

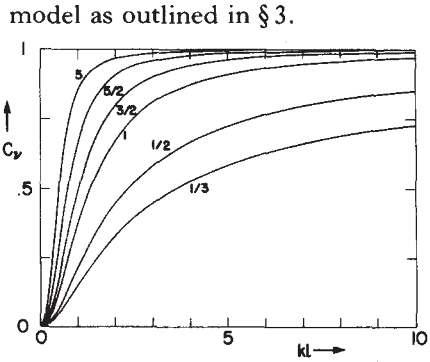

Figure 9. Plot of the radiation efficiency (6.8) for various values of ν.

For a given radiant intensity $I(\theta)$ or correlation $W(\rho)$, the underlying slope distribution $P(m)$ is, in principle, obtained by inversion of the relations (3.9) or (3.11), respectively. Such inversions are meaningful only if they lead to functions $P(m)$ with $P(m) \geqslant 0$ and $\int P(m) m\, dm < \infty$. Moreover, because of $|\sin\theta| \leqslant 1$, we can recover $P(m)$ only for $m \leqslant k\xi$. The formal application of relation (3.9) with (7.2) leads to the hypothetical 'lambertian' slope distribution

$$P_L(m) \propto (1 - m^2/k^2\xi^2)^{-1/2}, \quad m \leqslant k\xi \tag{7.4}$$

Apparently this is not a well behaved slope probability distribution.

Let us now reconsider the above question in terms of the more intuitive facet-angle distributions $\tilde{P}(\psi)$ with $\psi = \arctan m$ rather than slope distributions $P(m)$. The two distributions are related by

$$P(m) = \tilde{P}(\arctan m)\,(1 + m^2)^{1/2} \tag{7.5}$$

and

$$\tilde{P}(\psi) = P(\tan\psi)\cos\psi. \tag{7.6}$$

For example $\tilde{P}(\psi) = \text{const}$ leads to $P(m) \propto (1 + m^2)^{1/2}$ and the non-lambertian radiant intensity

$$I(\theta) \propto \cos^2\theta (1 + k^2\xi^2 \sin^2\theta)^{1/2} \tag{7.7}$$

whereas $P(m) = \text{constant}$ yields $\tilde{P}(\psi) \propto \cos\psi$ and $I(\theta) \propto \cos^2\theta$. For the lambertain scatterer we obtain the hypothetical distribution

$$\tilde{P}_L(\psi) \propto \cos\psi (1 - \tan^2\psi/k^2\xi^2)^{-1/2}, \quad \tan\psi < k\xi. \tag{7.8}$$

plotted in figure 10, with a singularity at $\psi \to \arctan(k\xi)$. We thus find that well behaved facet models with constant root mean square slope m_0 for strictly lambertian scatterers do not exist.

We finally recall that the obliquity factor $\cos^2\theta$ in equations (3.3) to (3.9) etc. is due to the scalar approximation adopted in this paper. In vector theory (for example [2, 30]) this factor is replaced by the expression $(1 + \cos^2\theta)/2$. With the latter factor the search for a lambertian facet model again does not lead to a well behaved slope distribution. For further discussion of the obliquity factor we refer to [31].

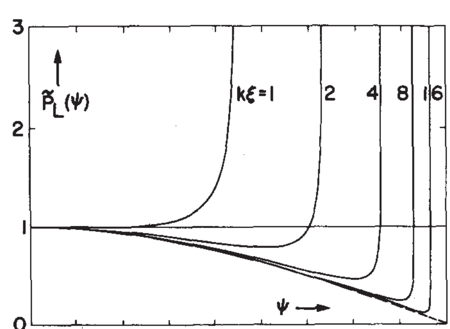

Figure 10. Hypothetical distributions of facet angles yielding lambertian radiant intensity. The $\cos\psi$ distribution leading to $\cos^2\theta$ radiant intensity is also shown (dashed curve).

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Die Intensitätsverteilung der gestreuten Strahlung über dem Streuwinkel kann für einen breiten Bereich von stochastischen Medien durch die Annahme von Quellen-Amplitudenkorrelationen einschließlich modifizierter Bessel-Funktionen K_ν erfaßt werden. Wir untersuchen, wie solche Korrelationen von physikalischen Modellen stochastischer Streusysteme, nämlich von Facettenflächen mit geeigneter Verteilung der Neigungen, abgeleitet werden können und betrachten die Fälle konstanter und variabler mittlerer quadratischer Neigung oder Facettengröße. Wir studieren die betreffende Photonenstatistik der an diesen Modellen gestreuten Strahlung und suchen eine Verbindung zwischen K-Korrelationen und K-Verteilungen. Als Anwendung betrachten wir die Konzepte und Gesetze der diffusen Reflexion und diskutieren die Existenz von Lambert-Streuern.

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